Problem Solving Sessions Solutions

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11 January 2025

Problem (1)

Let S be a non empty set with a binary operation $*$ defined on it. It has the following properties.

1. * is associative
2. it follows left cancellation i.e if $a * b = a * c$, then $b = c$
3. it follows right cancellation i.e. if $b * a = c * a$, then $b = c$
4. the set of powers of each element is finite i.e for all a in S, $\{a^1, a^2, a^3, a^4, ...\}$ is finite

Does it follow that S is a group?

Solution. For simplicity of notation, we don't use the $*$ symbol. Thus, we denote $a * b$ as just ab. Two of the group axioms are already given in the problem statement itself. So, we only need to prove the existence of identity and inverse.

Existence of Identity

Firstly, by the finiteness of the set of powers of a, it follows that there exists some m and n such that, $a^m = a^n$. Let $m - n = k$. Denote

$$e = a^k$$

We claim that e is the identity element. To prove this, first note that $a^{k+1} = a$. Now consider an arbitrary element g of S. Now consider the equation,

$$ga = ga^{k+1}$$

Cancelling a from both sides, we are done.

Existence of Inverse

Note that, for every element g in S, $g^t = 1$ for some t. The proof is very similar to how we did in the previous part. Thus, $gg^{t-1} = 1$, which forces that g^{t-1} is the inverse of g. $\square$

Problem (2)

Let n be a positive integer. Beto writes a list of n non-negative integers on the board. Then he performs a succession of moves (two steps) of the following type: First for each $i = 1, 2, ..., n$, he counts how many numbers on the board are less than or equal to i. Let a_i be the number obtained for each $i = 1, 2, ..., n$. Next, he erases all the numbers from the board and writes the numbers $a_1, a_2, ..., a_n$. For example, if $n = 5$ and the initial numbers on the board are $0, 7, 2, 6, 2$, after the first move, the numbers on the board will bec$1, 3, 3, 3, 3$;after the second move they will be $1, 1, 5, 5, 5$, and so on.

1. Show that, for every n and every initial configuration, there will come a time after which the numbers will no longer be modified when using this move.
2. Find (as a function of n) the minimum value of k such that, for any initial configuration, the moves made from move number k will not change the numbers on the board.

Argentina MO 2023

Solution. Call a configuration good if it produces itself under the operation described above. We claim that we can achieve the good configuration in at most $2n$ steps. For the construction, consider the sequence, $(1, n+1, \cdots, n+1)$, where there are $n-1$ terms which are $n+1$.

Claim: The only good sequence if $(1, 2, \cdots, n)$

Proof: We will prove this by induction on the terms. Firstly, we will prove that the first term has to be 1. Note that if the first term remains unchanged under the operation, then the term cannot be greater than 1, otherwise the term under the operation will get changed to 0. Also, the term cannot be 0, otherwise it gets changed to 1. This proves the base case. Now assume that the first k terms of the sequence are $1, 2, \cdots, k$ respectively. Let the $(k+1)$th term be t. If $t > k+1$, then we reach a contradiction by the same argument as above. If $t < k+1$, then again we reach a contradiction by the same logic as above. This ends the proof.

Now we will try to examine how many steps it takes to reach the good configuration. Note that by the end of move 2, the last term in the sequence has to be n. This is because in the first move itself, all the terms will change to something $\leq n$, because there can be at most n terms in the sequence. Now we claim by induction that by the move number $2k$, we have that the last terms of the sequence looks something like this, $(\cdots, n+1-k, n+2-k, \cdots, n)$. To prove this, it is enough to prove that it takes at most 2 steps to increase 1 more term at the end of the sequence to the form written above. To prove this, suppose we have the sequence something like this, $(a_1, a_2, \cdots, a_{t-1}, t, t+1, \cdots, n)$. Note that in the next step, no matter the values of a_i for any $i < t$, all the first $t-1$ terms will be changed to something $< t$. This is because there are already $n-t+1$ terms which are greater than $t-1$. Thus, after the second step, we are sure that there are exactly $t-1$ terms which are $\leq t-1$, and thus, the $t-1$th term in the sequence will become $t-1$. By induction, we are done.

□

Problem (3)

Prove that all positive integers n can be written as the difference of two positive integers a and b such that each of them have the same number of distinct prime divisors.

Sophie Fellowship

Solution. (Solution by Pradyun Yadav) We will consider two cases, when n is even and n is odd. If n is even, note that the pair $(n, 2n)$ satisfies the problem constraints, as multiplying an even number by 2 doesn't add any new prime factor. If n is odd, consider the smallest **odd** prime number that does not divide n. Then we claim that the pair $((p-1)n, pn)$ works. To prove this, firstly note that the number pn has exactly one more prime factor than n, namely p. Also, note that all the odd prime factors of $p-1$ already divide n, so they add no additional prime factor. However, 2 divides $p-1$, and hence adds one more prime factor. □

Problem (4)

100 rooms each contain countably many boxes labeled with the natural numbers. Inside of each box is a real number. For any natural number n, all 100 boxes labeled n (one in each room) contain the same real number. In other words, the 100 rooms are identical with respect to the boxes and real numbers.

Knowing the rooms are identical, 100 mathematicians play a game. After a time for discussing strategy, the mathematicians will simultaneously be sent to different rooms, not to communicate with one another again. While in the rooms, each mathematician may open up boxes (perhaps countably many) to see the real numbers contained within. Then each mathematician must guess the real number that is contained in a particular unopened box of his choosing. Notice this requires that each leaves at least one box unopened.

99 out of 100 mathematicians must correctly guess their real number for them to (collectively) win the game.

What is a winning strategy?

Predicting Real Numbers, StackExchange

Solution. (Typed By Pratham Gupta)

Problem Setup

- **Rooms and Boxes**: There are 100 rooms, each containing countably many boxes labeled by natural numbers n. The contents of the boxes labeled n are identical across all rooms.
- **Mathematician's Goal**: After strategizing together, 100 mathematicians are placed in separate rooms. Each can open some boxes and inspect the real numbers inside. Each mathematician must then guess the number inside one unopened box, ensuring that 99 out of the 100 guesses are correct.

Sequences and Representatives

- Each mathematician sees a *sequence* of real numbers by opening certain boxes. A sequence of numbers is considered as a whole.
- **Equivalence Relation on Sequences**: Two sequences are considered equivalent if they are equal after some finite index. For example, $(1, 2, 3, 4, \dots)$ and $(1, 2, 3, 4, 5, \dots)$ are equivalent because they differ only finitely.
- **Choice of Representatives**: The mathematicians agree beforehand to associate each equivalence class with a single *representative sequence*. This means any sequence they encounter can be "mapped" to its representative. **This step requires axiom of choice: ie their exist a choice function that selects a representative from each equivalence class.**

Strategy Details

1. **Pre-Game Agreement**:
    - Mathematicians agree to re-label boxes using pairs (m, x), where m identifies a mathematician, and x is a natural number.
        - This can be something like $(m, x) \equiv m + 100 \times x$
    - They also agree on representative sequences for equivalence classes.
2. **In the Room**:
    - Mathematician m opens every box **except those labeled** (m, x) **for** $x \in \mathbb{N}$.
    - For each mathematician $m' \neq m$, m examines the sequence $\big((m', x)\big)_{x\in\mathbb{N}}$.
        - He identifies the *largest index* $x(m')$ where this sequence differs from its representative.
        - If the sequence matches its representative entirely (ie. the sequence is the representative) then $x(m') = -1$.
    - Mathematician m calculates:

$$y(m) = \max_{m' \neq m} (x(m')) + 1$$

    - Then mathematician m opens all boxes labeled (m, x) for $x > y(m)$.
3. **Guessing**:
    - Mathematician m uses the sequence of his boxes (m, x) where $x > y(m)$ to find its representative. Based on this, m guesses the value in box $(m, y(m))$.

Why This Works

- For any m, if $y(m) > x(m')$ for all $m' \neq m$, then m's guess will be correct, as he has enough information to reconstruct the sequence.
- If one mathematician m has $x(m) > x(m')$ for all $m' \neq m$, then m may guess incorrectly. However, this happens to at most one mathematician, ensuring the success of the group as a whole (since 99 out of 100 guesses are correct).

Edge Cases

- **Tie in $x(m)$ Values**: If multiple mathematicians have the same largest $x(m)$, they all guess correctly because $y(m') > x(m')$ holds for them.

□

Problem

Aaron and Beren are playing a game on an infinite complete binary tree. At the beginning of the game, every edge of the tree is independently labeled A with probability p and B otherwise. Both players are able to inspect all of these labels. Then, starting with Aaron at the root of the tree, the players alternate turns moving a shared token down the tree (each turn the active player selects from the two descendants of the current node and moves the token along the edge to that node). If the token ever traverses an edge labeled B, Beren wins the game. Otherwise, Aaron wins.

What is the infimum of the set of all probabilities p for which Aaron has a nonzero probability of winning the game? Give your answer in exact terms.

Jane Street Puzzles

Solution. Suppose for a given p, Aaron's probability of winning on the infinite tree is x. Then, looking at the first two turns in the game, Aaron can win if at least one of the two sides of the tree has 3 edges marked A and **both** subtrees Beren can choose between are winnable by Aaron (which we know has probability x, independently per subtree). So the equation x and p must satisfy is:

$$x = 2p^3x^2 - p^6x^4.$$

(The subtracted term is from the double counting of when Aaron can win on both sides of the tree).

We want to find the smallest positive p where this has a positive root $x < 1$. This can be done with the theory of discriminants, but also can be solved by noticing that as p increases, the graph of the quartic that is the right-hand side of the above equation approaches the graph of $f(x) = x$ from below, and the moment where it touches it will be tangent. So we can add the constraint that the derivative of the right-hand side equals 1 at the point of equality:

$$1 = 2p^3(2x) - p^6(4x^3).$$

These two equations are solved by $x = \frac{8}{9}$, and $p = \left(\frac{27}{32}\right)^{1/3}$. So the lowest nonzero probability Aaron can have of winning the game is the surprisingly high $\frac{8}{9}$ chance! □

Problem

For each integer $n \geq 1$, compute the smallest possible value of

$$\sum_{k=1}^{n} \left\lfloor \frac{a_k}{k} \right\rfloor$$

over all permutations $(a_1, \ldots, a_n)$ of $\{1, \ldots, n\}$.

IMO Shortlist 2021 A3

Solution. (Credit: Biomathematics, read the original version on AOPS here) Let the smallest possible value be $F(n)$. We show that $F(n) = 1 + \lfloor \log_2 n \rfloor$ for all naturals n. Note that $F(1) = 1$. It suffices to show that for all naturals n, we have $F(2n) = F(n) + 1$ and $F(2n + 1) = F(n) + 1$. We show the former.

Suppose $\sigma \in \mathcal{S}_n$ achieves $\sum_{k=1}^{n} \left\lfloor \frac{\sigma(k)}{k} \right\rfloor = F(n)$. Extend σ to $\pi \in \mathcal{S}_{2n}$ as

$$\pi(k) = \begin{cases} \sigma(k) & 1 \leq k \leq n \\ 2n & k = n + 1 \\ k - 1 & n + 2 \leq k \leq 2n \end{cases}$$

Then

$$\sum_{k=1}^{n} \left\lfloor \frac{\pi(k)}{k} \right\rfloor = F(n) + \left\lfloor \frac{2n}{n+1} \right\rfloor + \sum_{k=n+2}^{2n} \left\lfloor \frac{k-1}{k} \right\rfloor = F(n) + 1,$$

so $F(2n) \leq F(n) + 1$.

Suppose $F(2n) \leq F(n)$. Consider a permutation $\pi \in \mathcal{S}_{2n}$ such that $\sum_{k=1}^{2n} \left\lfloor \frac{\pi(k)}{k} \right\rfloor = F(2n)$. Consider the first n elements $\pi(1), \pi(2), \ldots, \pi(n)$. The m-th smallest number among these is at least m, for each $1 \leq m \leq n$. Define $\sigma \in \mathcal{S}_n$ as $\sigma(k) = m$ if $\pi(k)$ is the m-th smallest among $\pi(1), \pi(2), \ldots, \pi(n)$. Then,

$$F(n) \geq F(2n) = \sum_{k=1}^{2n} \left\lfloor \frac{\pi(k)}{k} \right\rfloor \geq \sum_{k=1}^{n} \left\lfloor \frac{\pi(k)}{k} \right\rfloor \geq \sum_{k=1}^{n} \left\lfloor \frac{\sigma(k)}{k} \right\rfloor \geq F(n)$$

So equality holds throughout. This means $\left\lfloor \frac{\pi(k)}{k} \right\rfloor = 0$ for $n + 1 \leq k \leq 2n$ and $\left\lfloor \frac{\pi(k)}{k} \right\rfloor = \left\lfloor \frac{\sigma(k)}{k} \right\rfloor$ for $1 \leq k \leq n$. But where does the number $2n$ go?

Note that $\left\lfloor \frac{2n}{k} \right\rfloor \geq 1$ for any $n + 1 \leq k \leq 2n$, and if $\pi(k) = 2n$ for some $1 \leq k \leq n$, then $\left\lfloor \frac{\pi(k)}{k} \right\rfloor > \left\lfloor \frac{\sigma(k)}{k} \right\rfloor$, because the difference $\frac{\pi(k)}{k} - \frac{\sigma(k)}{k} = \frac{2n}{k} - \frac{n}{k} \geq 1$. This is a contradiction.

Therefore $F(2n) = F(n) + 1$. The proof of $F(2n + 1) = F(n) + 1$ is similar. $\square$