Math Problem Solving Session 1 - Solutions

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Problem 1

Suppose the first n prime numbers $2, 3, 5, \ldots p_n$, are partitioned into two sets arbitrarily. Let P_1 and P_2 be the products of all the elements in set 1 and set 2 respectively. In case of an empty set, let $P_i = 1$. Prove that each of the numbers $P_1 + P_2$ and $|P_1 - P_2|$ will also always turn out to be a prime number for every such partition, given that P_i is less than p_{n+1}^2

Remark. The problem in its current form was not the intended problem. The intended problem had the condition that $|P_1 \pm P_2| < p_{n+1}^2$. Though I think that the given problem is also true, it would require much more work in proving it. One argument involves proving that $p_1 p_2 \cdots p_n \geq p_{n+1}^4 \ \forall n \geq N_0$ for some N_0 and individually checking the validity of the given statement for $n < N_0$. So you may consider the given problem as a challenge problem to prove or disprove. I present the intended problem and the solution to the intended problem.

Intended Problem

Suppose the first n prime numbers $2, 3, 5, \ldots p_n$, are partitioned into two sets arbitrarily. Let A and B be the partitioned sets and let P_A and P_B be the products of all the elements in A and B respectively. In case either A or B is empty, let P_A or $P_B = 1$. Prove that each of the numbers $P_1 + P_2$ and $|P_1 - P_2|$ will also always turn out to be a prime number for every such partition, given that $|P_A \pm P_B|$ is less than p_{n+1}^2

(Source : More Mathematical Morsels)

Solution

Each of the first n prime numbers divide either P_A or P_B but not both. So none of them divide $P_A + P_B$ or $|P_A - P_B|$. Then, any prime divisor of $P_A + P_B$ or $|P_A - P_B|$ is greater than or equal to p_{n+1}. If there were more than one prime divisor, then the number would be greater than or equal to

p_{n+1}^2 which is not possible. Therefore the number itself must be a prime p such that $p_{n+1} \leq p < p_{n+1}^2$

Problem 2

Determine all functions $f : \mathbb{N} \mapsto \mathbb{N}$ satisfying

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all positive integers x and y.

(Source : 2002 Canadian Mathematical Olympiad)

Solution 1

The constant function $f(x) = k$, where k is a positive integer, is the only possible solution. That any such function satisfies the given condition is easy to check.
Now suppose there exists a nonconstant solution f . There must exist two positive integers a and b such that $f(a) < f(b)$. This implies that

$$(a + b)f(a) < af(b) + bf(a) < (a + b)f(b)$$

By the given condition this is equivalent to

$$(a + b)f(a) < (a + b)f(a^2 + b^2) < (a + b)f(b)$$

We can divide by $a + b > 0$ to obtain that

$$f(a) < f(a^2 + b^2) < f(b)$$

Thus between any two different values of f we can insert another. But this cannot go on forever, since f takes only integer values. The contradiction shows that such a function cannot exist. Thus constant functions are the only solutions.

Solution 2

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2) \tag{1}$$

Substituting $y = x$ in 1, we get

$$\begin{aligned} 2xf(x) &= (2x)f(2x^2) \\ f(x) &= f(2x^2) \qquad \text{(as } x \neq 0) \end{aligned}$$

Substituting $x = 1$ in 1, we get

$$xf(1) + f(x) = (x + 1)f(x^2 + 1)$$

So we have

$$xf(1) + f(x) \equiv 0 \pmod{(x+1)}$$

Since $x \equiv -1 \pmod{(x+1)}$

$$-f(1) + f(x) \equiv 0 \pmod{(x+1)}$$

Therefore

$$|f(x) - f(1)| \equiv 0 \pmod{(x+1)} \tag{2}$$

Replacing x by $2x^2$, we get

$$|f(2x^2) - f(1)| \equiv 0 \pmod{(2x^2+1)}$$

Since $f(x) = f(2x^2)$, we have

$$|f(x) - f(1)| \equiv 0 \pmod{(2x^2+1)}$$

Let $a \in \mathbb{N}$ be a constant and let $f(a) = b$. Now using $f(x) = f(2x^2)$, we can obtain

$$b = f(a) = f(2a^2) = f(8a^4) = f(128a^8)\ldots$$

From 2, we have

$$|f(a) - f(1)| \equiv 0 \pmod{(k+1)} \qquad k = f(a), f(2a^2), f(8a^4)\ldots$$

Thus $|f(a) - f(1)|$ has infinitely many divisors. So $f(a) - f(1) = 0$. Since a was an arbitrary natural number and $f(1)$ is a constant, we can conclude that $f(x)$ is constant for all postive integers x

Problem 3

You are the manager of a stadium, which is heavily overbooked. Your goal is to devise a method such that every person who shows up to the stadium has equal probability of being seated as compared to any other person (regardless of the time at which this person shows up) but you also don't want any seat to remain empty. How will you do it? Once a person is refused entry, he/she will never come back.

Alternate Problem Statement

Given a stream of input data where the total number of data points is unknown and with limited space to store data, provide an algorithm such that at the end of the stream, each of the data points has an equal probability of being selected

(Source : The Idea of Reservoir Sampling)

Solution 3

Initialize an array R indexed from 1 to k, containing the first k items of the input $x_1, \ldots, x_k$. This is the reservoir.

For each new input x_i, generate a random number j uniformly in $\{1, \ldots, i\}$. If $j \in \{1, \ldots, k\}$, then set $R[j] := x_i$. Otherwise, discard x_i.

Return R after all inputs are processed.

This algorithm works by induction on $i \geq k$.

Proof of the algorithm

When $i = k$, Algorithm R returns all inputs, thus providing the basis for a proof by mathematical induction.

Here, the induction hypothesis is that the probability that a particular input is included in the reservoir just before the $(i+1)$-th input is processed is $\frac{k}{i}$ and we must show that the probability that a particular input is included in the reservoir is $\frac{k}{i+1}$ just after the $(i+1)$-th input is processed.

Apply Algorithm R to the $(i+1)$-th input. Input x_{i+1} is included with probability $\frac{k}{i+1}$ by definition of the algorithm. For any other input $x_r \in \{x_1, \ldots, x_i\}$, by the induction hypothesis, the probability that it is included in the reservoir just before the $(i+1)$-th input is processed is $\frac{k}{i}$. The probability that it is still included in the reservoir after x_{i+1} is processed (in other words, that x_r is not replaced by x_{i+1} is $\frac{k}{i} \times \left(1 - \frac{1}{i+1}\right) = \frac{k}{i+1}$.

The latter follows from the assumption that the integer j is generated uniformly at random; once it becomes clear that a replacement will in fact occur, the probability that x_r in particular is replaced by x_{i+1} is $\frac{k}{i+1}\frac{1}{k} = \frac{1}{i+1}$.

We have shown that the probability that a new input enters the reservoir is equal to the probability that an existing input in the reservoir is retained. Therefore, we conclude by the principle of mathematical induction that Algorithm R does indeed produce a uniform random sample of the inputs.

Problem 4

The "board" on which this 2-person game is played is a monic polynomial

$$f(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + 1$$

in which the $2n-1$ coefficients $a_{2n-1}, \ldots, a_1$ are unspecified real numbers to begin with. The degree is set at any even integer ≥ 4, and the players, starting with A, take turns specifying an undetermined coefficient until $f(x)$ is completely defined. At the end, A wins if no root of $f(x) = 0$ is real, and B otherwise. Determine a winning strategy for B.

(Source : More Mathematical Morsels)

Solution 4

Since the degree is even, there is always an odd number of unspecified terms at the beginning of the game, one more term of odd degree than of even degree. For example, for $2n = 10$, there are 9 unspecified terms, 5 of odd degree $\left(x, x^3, x^5, x^7, x^9\right)$ and 4 of even degree $\left(x^2, x^4, x^6, x^8\right)$. As we shall see, it is to B 's advantage to preserve this delicate imbalance, and so B 's initial strategy is simply to restore the status quo by doing just the opposite of what A does- if A specifies a term of odd degree. then B specifies one of even degree, and vice versa. At this point it doesn't matter what value either A or B may assign to his choice of coefficient. B stays with this approach until there are exactly 3 terms left to be specified. At this stage, then, there must be left two terms of odd degree and one of even degree. Since there is an odd number of unspecified coefficients to begin with, it is again A's turn to play when there are just 3 terms left, and, after making this move, he must present to B a polynomial with only 2 unspecified terms, ax^s and bx^t, where either (i) one of s, t is even and the other odd, or (ii) both s and t are odd.
If $P(x)$ denotes the part of $f(x)$ that has already been specified to this point, then $f(x) = P(x) + ax^2 + bx^t$, where a and b are still to be specified. Let us consider the two cases separately. (i) Suppose s is even and t is odd. In this case,

$$f(1) = P(1) + a + b$$
$$f(-1) = P(-1) + a - b$$

giving

$$f(1) + f(-1) = P(1) + P(-1) + 2a.$$

Now B wins by choosing the value of a so as to make the right side of this equation equal to zero:

$$a = -\frac{P(1) + P(-1)}{2}$$

This makes $f(1) + f(-1) = 0$, no matter what A puts for the value of b on the final move of the game. Therefore either (a) $f(1)$ and $f(-1)$ are themselves both zero, making both $x = 1$ and $x = -1$ real roots, or (b) one of $f(1), f(-1)$ is negative and the other positive, forcing a real root between 1 and -1 (by the continuity of polynomials). (ii) Suppose s and t are both odd. In this case, $f(1)$ and $f(-1)$ contain the same function of a and b, and cannot be solved for a and b. However, if $f(1)$ is replaced by $f(2)$, we obtain

$$\begin{aligned} f(-1) &= P(-1) - a - b \\ f(2) &= P(2) + a2^x + b2^t \end{aligned}$$

Thus b is eliminated by multiplying $f(-1)$ by 2^t and adding:

$$2^t f(-1) + f(2) = 2^t \cdot P(-1) + P(2) - a\left(2^t - 2^s\right)$$

Again, then, B crushes A by choosing a so as to make the right side vanish:

$$a = \frac{2^t P(-1) + P(2)}{2^t - 2^s}$$

($s \neq t$ since they are exponents of different unspecified terms.) This makes $2^t f(-1) + f(2) = 0$, no matter how A may specify the last coefficient b and again either (a) both $f(-1)$ and $f(2)$ are themselves zero, or (b) they straddle zero (the factor 2^t doesn't affect this), implying a real root in either case.

Problem 5

You are playing a game with the Devil. There are n coins in a line, each showing either H (heads) or T (tails). Whenever the rightmost coin is H, you decide its new orientation and move it to the leftmost position. Whenever the rightmost coin is T , the Devil decides its new orientation and moves it to the leftmost position. If you can make all coins face the same way within 2^ns, you win, else the devil wins. Show that you have a winning strategy, i.e., no matter what the devil does, you can find a way to make all coins heads or all coins tails.

(Source : Some Online Math Club *session)*

Solution 5

We will first show that it is possible to achieve the desired configuration, and worry about the number of steps later. Firstly, replace the heads with 0 and the tails with 1. Then we do yet another simplification. Rather than moving the coin itself to the left most end after each turn, what we do is imagine we have a box placed at the coin where we will perform the operation. For the binary string 1001, this looks something like this,

$$100\boxed{1} \to 10\boxed{0}1 \to 1\boxed{0}01 \to \boxed{1}001$$

Where the boxed place is where the operation is taking place. After all the coins have been acted upon, the box again moves to the rightmost end. Mark this as the end of a *round*. It is clear that this setup is equivalent to the problem statement. Our strategy is to change all the 0's to 1, *until* the devil changes a 1 to a 0. To prove that this strategy works, note that if the devil changes no 1 to a 0, then we win because we get all 1's. If the devil changes any of the 1's to a 0, after that, we will keep the 0's left to that place as a 0. Note that if we follow this strategy, either we win at the end of a *round* or, the binary value *strictly decreases*. It is easy to show that this strategy terminates with at most 2^n steps, and we leave it as an exercise!

Problem 6

Let $\mathcal{F}$ be the set of differentiable functions $f : [0,1] \to \mathbb{R}$, with f' continuous over $[0,1]$ and which verify the equalities

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 1$$

Find $\min\left\{\int_0^1 (f'(x))^2\, dx \mid f \in \mathcal{F}\right\}$

(Source : Gazeta Mathematica Vol.2022,No.3-4)

Solution 6

Solution. Using integration by parts, we obtain that

$$1 = \int_0^1 f(x)\mathrm{d}x = xf(x)|_0^1 - \int_0^1 xf'(x)\mathrm{d}x = f(1) - \int_0^1 xf'(x)\mathrm{d}x$$

and

$$1 = \int_0^1 xf(x)\mathrm{d}x = \frac{1}{2}x^2 f(x)\Big|_0^1 - \frac{1}{2}\int_0^1 x^2 f'(x)\mathrm{d}x = \frac{f(1)}{2} - \frac{1}{2}\int_0^1 x^2 f'(x)\mathrm{d}x$$

By canceling out $f(1)$ in the two equalities, we get that $\int_0^1 \left(x - x^2\right) f'(x)\mathrm{d}x =$ The Cauchy-Schwarz inequality then leads to

$$\begin{aligned} 1 = \left(\int_0^1 \left(x - x^2\right) f'(x)\mathrm{d}x\right)^2 &\leq \int_0^1 \left(x - x^2\right)^2 \,\mathrm{d}x \cdot \int_0^1 \left(f'(x)\right)^2 \,\mathrm{d}x \\ &= \frac{1}{30}\int_0^1 \left(f'(x)\right)^2 \,\mathrm{d}x \end{aligned}$$

with equality if and only if $f'(x) = \alpha\left(x - x^2\right)$ for some $\alpha \in \mathbb{R}$, since f' is continuous. This happens when $f(x) = \alpha\left(\frac{x^2}{2} - \frac{x^3}{3}\right) + \beta$, where the constants α

and β are such that $f \in \mathcal{F}$. A simple computation leads to the linear system

$$\begin{cases} \dfrac{1}{12}\alpha + \beta = 1 \\ \dfrac{7}{120}\alpha + \dfrac{1}{2}\beta = 1 \end{cases}$$

with the solution $\alpha = 30, \beta = -\frac{3}{2}$. Concluding, $\min\left\{\int_0^1 \left(f'(x)\right)^2 \, \mathrm{d}x \mid f \in \mathcal{F}\right\} = 30$, which is achieved for the function $f(x) = -10x^3 + 15x^2 - \frac{3}{2}$.

Remark. There was another solution proposed using Euler Lagrange equation which turned out to be incorrect. The reason for incorrectness was the fact that in the derivation of Euler Lagrange equation the end points are fixed and the function is varied. But here the initial conditions given by the the two equations did not imply fixed values at boundary of $f(x)$.