Problem Solving Session

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Problem 1

Let S be a non empty set with a binary operation $*$ defined on it. It has the following properties.

1. * is associative
2. it follows left cancellation i.e if $a * b = a * c$, then $b = c$
3. it follows right cancellation i.e. if $b * a = c * a$, then $b = c$
4. the set of powers of each element is finite i.e for all a in S, $\{a^1, a^2, a^3, a^4, ...\}$ is finite

Does it follow that S is a group? (See `https://proofwiki.org/wiki/Axiom:Group_Axioms` for group axioms. Note that you don't need prior knowledge of group theory to attempt this)

Curated by Udit Shah

Problem 2

Let n be a positive integer. Beto writes a list of n non-negative integers on the board. Then he performs a succession of moves (two steps) of the following type: First for each $i = 1, 2, ..., n$, he counts how many numbers on the board are less than or equal to i. Let a_i be the number obtained for each $i = 1, 2, ..., n$. Next, he erases all the numbers from the board and writes the numbers $a_1, a_2, ..., a_n$. For example, if $n = 5$ and the initial numbers on the board are $0, 7, 2, 6, 2$, after the first move, the numbers on the board will bec$1, 3, 3, 3, 3$;after the second move they will be $1, 1, 5, 5, 5$, and so on.

1. Show that, for every n and every initial configuration, there will come a time after which the numbers will no longer be modified when using this move.
2. Find (as a function of n) the minimum value of k such that, for any initial configuration, the moves made from move number k will not change the numbers on the board.

Curated by Shreeansh Hota

Problem 3

Prove that all positive integers n can be written as the difference of two positive integers a and b such that each of them have the same number of distinct prime divisors.

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Problem 4

100 rooms each contain countably many boxes labeled with the natural numbers. Inside of each box is a real number. For any natural number n, all 100 boxes labeled n (one in each room) contain the same real number. In other words, the 100 rooms are identical with respect to the boxes and real numbers.

Knowing the rooms are identical, 100 mathematicians play a game. After a time for discussing strategy, the mathematicians will simultaneously be sent to different rooms, not to communicate with one another again. While in the rooms, each mathematician may open up boxes (perhaps countably many) to see the real numbers contained within. Then each mathematician must guess the real number that is contained in a particular unopened box of his choosing. Notice this requires that each leaves at least one box unopened.

99 out of 100 mathematicians must correctly guess their real number for them to (collectively) win the game.

What is a winning strategy?

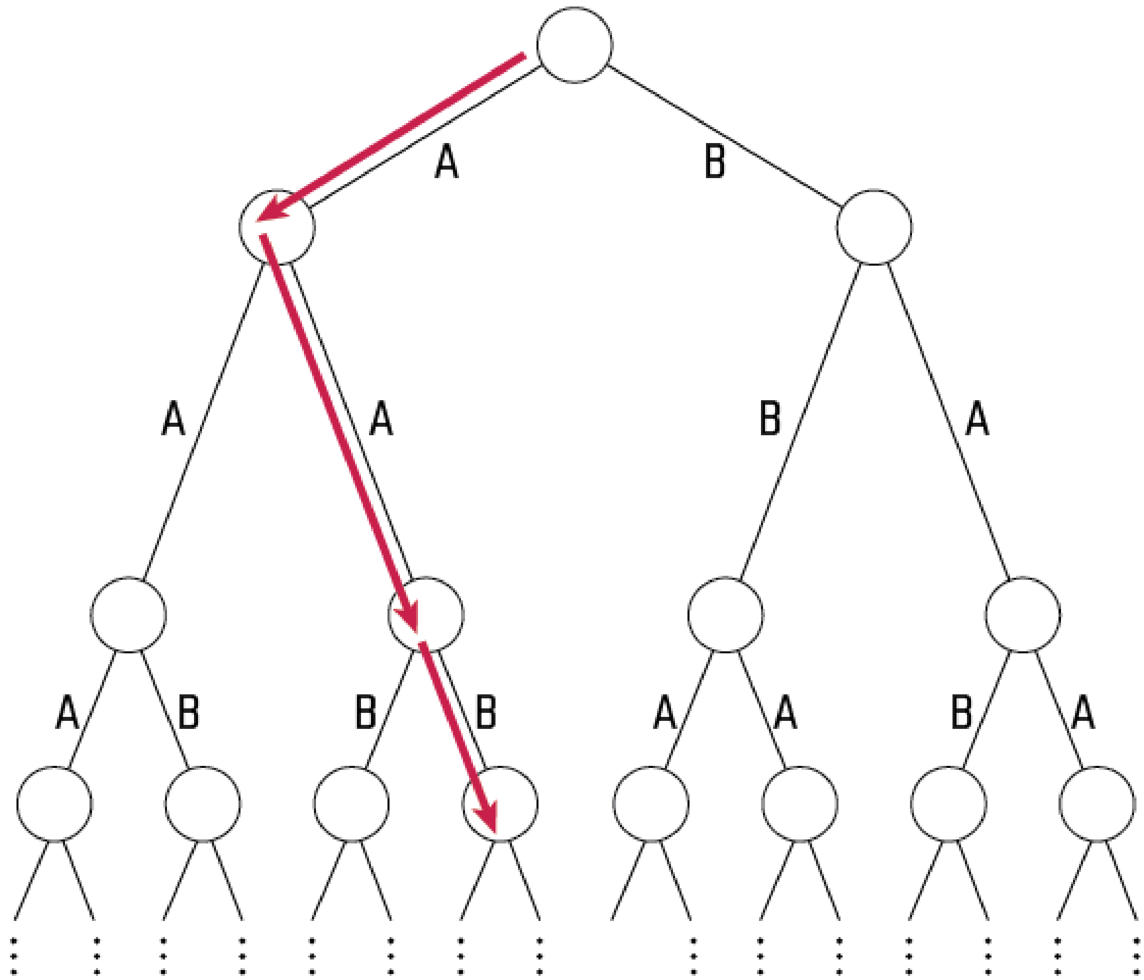
Curated by Pratham Gupta

Problem 5

Aaron and Beren are playing a game on an infinite complete binary tree. At the beginning of the game, every edge of the tree is independently labeled A with probability p and B otherwise. Both players are able to inspect all of these labels1. Then, starting with Aaron at the root of the tree, the players alternate turns moving a shared token down the tree (each turn the active player selects from the two descendants of the current node and moves the token along the edge to that node). If the token ever traverses an edge labeled B, Beren wins the game. Otherwise, Aaron wins.

An example game is in the picture below: after the labeling, Aaron chooses to go left to avoid immediate defeat, but after Beren goes right Aaron is doomed to choose one of two B paths and Beren wins.

What is the infimum of the set of all probabilities p for which Aaron has a nonzero probability of winning the game? Give your answer in exact terms.



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Bonus Problem

For each integer $n \geq 1$, compute the smallest possible value of

$$\sum_{k=1}^{n} \left\lfloor \frac{a_k}{k} \right\rfloor$$

over all permutations $(a_1, \ldots, a_n)$ of $\{1, \ldots, n\}$.

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