



Problem Solving Session 4

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Problem 1

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers with the property that for any $n \geq 2$ there exists an integer $k \in [\frac{n}{2}, n)$ such that $a_n = \frac{1}{2}a_k$. Prove that $\lim_{n \rightarrow \infty} a_n = 0$.

Problem 2

A graph with $2n$ vertices has $n^2 + 1$ edges. Show that the graph has a triangle.

Problem 3

Pravega's legal code specifies as a holiday any day on which at least one worker has a birthday. All other days are working days. How many unpaid workers must Pravega employ to maximize the number of working man-days in a year? (2 workers working for 3 days is 6 working man-days.)

Problem 4

Prof. Dumbledore was flaunting his collection of magic squares to Hogwarts' students: $n \times n$ grids of rational numbers with

- (Case 1) each row and each column summing up to the same number.
- (Case 2) each row, each column and the principal diagonal (top-left to bottom-right) summing up to the same number.
- (Case 3) each row, each column and both diagonals summing up to the same number.

The students were hungry for more, so he came up with a way to quickly generate them. For each case, how many magic squares would Dumbledore need so that any magic square can be generated by their linear combinations?

Hermione found only integral examples exciting. Would your answer change if each entry was required to be an integer?

Problem 5

Prove that for any even natural number n , $2^{n!} - 1$ is a multiple of $n^2 - 1$.

Problem 6

For every natural number n , show that there is a circle in the plane which contains exactly n lattice points in its interior. (Lattice points are points whose coordinates are integers.)