Problem Solving Session 3

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Problem 1:

Consider all lattice squares (x, y) with x, y nonnegative integers. Assign to each its lower left corner as a label. We shade the squares $(0, 0)$, $(1, 0)$, $(0, 1)$, $(2, 0)$, $(1, 1)$, $(0, 2)$.

(a) There is a chip on each of the six squares
(b) There is only one chip on $(0, 0)$.

Step: If (x, y) is occupied, but $(x + 1, y)$ and $(x, y + 1)$ are free, you may remove the chip from (x, y) and place a chip on each of $(x + 1, y)$ and $(x, y + 1)$. The goal is to remove the chips from the shaded squares. Is this possible in the cases (a) or (b)?

Problem 2:

A device generates a random number uniformly distributed on $[0, 1]$ each time a button is pressed. Numbers are generated sequentially until their cumulative sum surpasses 1. Determine:

1. The probability of requiring more than n button presses.
2. The expected number of button presses needed.

Problem 3:

On a strange railway line, there is just one infinitely long track, so overtaking is impossible. Any time a train catches up to the one in front of it, they link up to form a single train moving

1. at the speed of the slower train
2. at the speed of the faster train
3. at the average speed of the two trains.

At first, there are n equally spaced trains, each moving at a different speed. You watch, and eventually (after all the linking that will happen has happened), you count the trains. You wonder what would have happened if the trains had started in a different order (but each of the original three trains had kept its same starting speed). On average (averaging over all possible orderings), how many trains will there be after a long time has elapsed?

Problem 4:

Find all integers n satisfying $n \geq 2$ and $\dfrac{\sigma(n)}{p(n)-1} = n$, in which $\sigma(n)$ denotes the sum of all positive divisors of n, and $p(n)$ denotes the largest prime divisor of n.

Problem 5:

A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called historic if $\{z-y, y-x\} = \{1776, 2001\}$. Show that the set of all nonnegative integers can be written as the union of pairwise disjoint historic sets.

Problem 6:

Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f(x + y^2 f(x^2)) = f(xy)^2 + f(x)$$

for all $x, y \in \mathbb{R}^+$.

Bonus Problem

Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \dots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \ldots a_n}$$

converges, and evaluate S.[1]

[1]Our dear friend Udit spent a lot of time typing out solutions for the first problem set. Send him some love and hearts guys!