



Problem Solving Session 2

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Problem 1:

In some country all roads between cities are one-way and such that once you leave a city you cannot return to it again. Prove that there exists a city into which all roads enter and a city from which all roads exit.

Problem 2:

Let x, y, z be positive integers such that $xy - z^2 = 1$. Prove that there exist nonnegative integers a, b, c, d such that

$$x = a^2 + b^2, \quad y = c^2 + d^2, \quad z = ac + bd.$$

Problem 3:

Consider the sequence of polynomials defined by $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, 3, \dots$. Show that for every positive integer n , the roots of the equation $P_n(x) = x$ are all real and distinct.

Problem 4:

Let m and n be natural numbers. Prove that in any sequence of length $mn + 1$ with all different real numbers, there is an increasing subsequence of length $m + 1$ or a decreasing subsequence of length $n + 1$.

Problem 5:

Does there exist an infinite sequence of integers $a_0, a_1, a_2, \dots$ such that $a_0 \neq 0$ and, for any integer $n \geq 0$, the polynomial

$$P_n(x) = \sum_{k=0}^n a_k x^k$$

has n distinct real roots?

*Problem 6:

I toss a sphere and color in red the hemisphere that is facing directly up when it stops. I continue to do this (potentially coloring over parts already colored in red). What is the expected number of toss until the whole sphere is red?