Problem Solving Session 1

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Problem 1: Prime Partitioning

Suppose the first n prime numbers $2, 3, 5, \ldots, p_n$, are partitioned into two sets arbitrarily. Let P_1 and P_2 be the products of all the elements in set 1 and set 2 respecticely. In case of an empty set, let $P_i = 1$.
Prove that each of the numbers $P_1 + P_2$ and $|P_1 - P_2|$ will also always turn out to be a prime number for every such partition, Given that P_i is less than p_{n+1}^2.
For example:
(i) For $(2, 3, 5)$: $(p_{n+1}^2 = 7^2 = 49)$

$$\begin{aligned} &2 \cdot 3 + 5 = 11, && 2 \cdot 5 + 3 = 13, && \\ &2 \cdot 5 - 3 = 7, && 3 \cdot 5 + 2 = 17, && \\ &3 \cdot 5 - 2 = 13, && 2 \cdot 3 \cdot 5 + 1 = 31, && 2 \cdot 3 \cdot 5 - 1 = 29. \end{aligned}$$

Problem 2: Functional Equation

Determine all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all positive integers x and y.

Problem 3: Stressful Job

You are manager of a Stadium, which is heavly overbooked. Your Goal to devise a method such that every person who shows up to the stadium has equal probability of being seated as compared to any other person (Regardless of the time at which this person shows up) but you also don't want any seat to remain empty. How will you do it? Once a person is refused entry, he/she will never come back.

Problem 4: Puzzling Polynomial

The "board" on which this 2-person game is played is a monic polynomial

$$f(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + 1,$$

in which the $2n-1$ coefficients $a_{2n-1}, \ldots, a_1$ are unspecified real numbers to begin with. The degree is set at any even integer ≥ 4, and the players, starting with A, take turns specifying an undetermined coefficient until $f(x)$ is completely defined. At the end, A wins if no root of $f(x) = 0$ is real, and B otherwise.

Determine a winning strategy for B.

Problem 5: Game with the Devil

You are playing a game with the Devil. There are n coins in a line, each showing either H (heads) or T (tails). Whenever the rightmost coin is H, you decide its new orientation and move it to the leftmost position. Whenever the rightmost coin is T, the Devil decides its new orientation and moves it to the leftmost position. If you can make all coins face the same way within 2^n moves, you win - else the devil wins. Show that you have a winning strategy, i.e., no matter what the devil does, you can find a way to make all coins heads or all coins tails.

Problem 6: Function Fitness

Let $\mathcal{F}$ be the set of differentiable functions $f : [0,1] \to \mathbb{R}$, with f' continuous over $[0,1]$ and which verify the equalities

$$\int_0^1 f(x)\,dx = \int_0^1 x f(x)\,dx = 1.$$

Find $\min \left\{ \int_0^1 (f'(x))^2\,dx \;\middle|\; f \in \mathcal{F} \right\}$.